Math 275D Lecture 7 Notes

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1 Equality of σ-Fields and Brownian Inversion

1.1 $\mathcal{F}_s^0$ and $\mathcal{F}_s^+$ are almost the same

Last time, we showed the Markov property for Brownian motion:

$$\mathbb{E}_x[Y \circ \theta_s \mid \mathcal{F}_s^+] = \mathbb{E}_{B(s)}[Y].$$

This is actually a bit stronger than a Markov property, since it uses $\mathcal{F}_s^+$, not $\mathcal{F}_s^0$.

Proposition 1.1. *$\mathcal{F}_s^+ = \mathcal{F}_s^0$ modulo null sets.*

Proof. We claim that $\mathbb{E}_x[Y \circ \theta_s \mid \mathcal{F}_s^0] = \mathbb{E}_{B(s)}[Y]$. The right hand side is $\mathcal{F}_s^0$-measurable, and the Markov property shows that it satisfies the definition of the conditional expectation. Then for any $\mathcal{F}$-measurable Z,

$$\mathbb{E}[Z \mid \mathcal{F}_s^+] = \mathbb{E}[Z \mid \mathcal{F}_s^0].$$

This follows from the monotone class argument, which tells us we only need to show it for $Z = \prod_{i=1}^k f(B(t_i))$. We can assume that $t_1 < t_2 < \cdots < r_m \leq s$ and $t_{m+1} > \cdots > t_k > s$. Then $Z = X \cdot (Y \circ \theta_s)$, where $X = \prod_{i=1}^m f(B(t_i))$ and $Y = \prod_{j=1}^{k-m} f(B(t_j - s))$. Then X is $\mathcal{F}_s^0$-measurable, so

$$\begin{aligned}
\mathbb{E}[Z \mid \mathcal{F}_s^0] &= \mathbb{E}[X(Y \circ \theta_s) \mid \mathcal{F}_s^0] \\
&= X\,\mathbb{E}[Y \circ \theta_s \mid \mathcal{F}_s^0] \\
&= X\,\mathbb{E}[Y \circ \theta_s \mid \mathcal{F}_s^+] \\
&= \mathbb{E}[X(Y \circ \theta_s) \mid \mathcal{F}_s^+] \\
&= \mathbb{E}[Z \mid \mathcal{F}_s^+].
\end{aligned}$$

□

1.2 $tB(1/t)$ is a Brownian motion

Last time, we mentioned the following property.

Proposition 1.2. *Let $Y(t) = tB(1/t)$. Then $Y(t)$ is a Brownian motion.*

Proof. $(Y(t_1), \dots, Y(t_n))$ is a Gaussian random vector. So to prove that $Y(t_2) - Y(t_1) \perp Y_(t_4) - Y(t_3)$, for example, we only need to prove that they are uncorrelated. It now remains to show that we can define $Y(0) = 0$. $\square$

We need to know that $\lim_{t\to\infty} \frac{B(t)}{t} = 0$ a.s.

Proposition 1.3. $\lim_{n\to\infty} \frac{B(n)}{n} = 0$. *a.s.*

Proof. $B(n) = \sum_{i=1}^n X_i$, where $X_i = B(n) - B(n-1)$. The X_i are iid with $N(0,1)$ distribution, so the strong law of large numbers gives the result. $\square$

What if we want to find the following probability:

$$\mathbb{P}\left(\max_{m\in[n,n+1]} \frac{|B(m) - B(n)|}{n^{2/3}} \geq 1 \right).$$

We can try looking at the following:

$$\mathbb{P}\left(\max_{m\in n+\mathbb{Q}_{[0,1]}^{(k)}} \frac{|B(m) - B(n)|}{n^{2/3}} \geq 1 \right),$$

where $\mathbb{Q}_{[0,1]}^{(k)} = \{\ell/k \in [0,1] : k, \ell \in \mathbb{Z}\}$

You could try a union bound:

$$\leq \sum_{\ell=1}^{k} \mathbb{P}\left(\left| \frac{B(n+\ell/k) - B(n)}{n^{2/3}} \right| \geq 1 \right).$$

However, this probability does not decay with k, and we have to add together k of them. So this will not work.

Let $X_\ell = B(n+\ell/k) - B(n+(\ell-1)/k)$, and let $Y_\ell = \sum_{\ell'<\ell} X_{\ell'}$. The X_ℓs are iid, so Y_ℓ is a Markov chain and a Martingale. We have the general inequality:

$$\mathbb{P}\left(\max_{1\leq\ell\leq k} |Y_\ell| \geq a \right) \leq \frac{\mathbb{E}[Y_k^2]}{a^2}$$

This gives us

$$\mathbb{P}\left(\max_{1\leq\ell\leq k} |Y_\ell| \geq n^{2/3} \right) \leq \frac{1}{n^{4/3}}.$$

Let $k = 2^{\tilde{k}}$, and define the event $A_{\tilde{k}} = \{\max_{m \in n+\mathbb{Q}_k} |B(m) - B(n)| \leq n^{2/3}\}$. Then $A_{\tilde{k}} \supseteq A_{\tilde{k}+1}$. We also have that

$$\mathbb{P}(A_{\tilde{k}+1}) \geq 1 - n^{-4/3}.$$

So

$$\mathbb{P}\left(\bigcup_{\tilde{k}} A_{\tilde{k}}\right) \geq 1 - n^{-4/3}.$$

This gives us

$$\mathbb{P}\left(\max_{m \in n+[0,1]} \left|\frac{B(m) - B(n)}{n^{2/3}}\right| \geq 1\right) \leq n^{-4/3}.$$

If we call this event C_n, we get that

$$\mathbb{P}(C_n \text{ i.o.}) = 0$$

by the first Borel-Cantelli lemma.

Together with the fact that $\lim_n \frac{B(n)}{n} \to 0$, we get:

Proposition 1.4. *With probability 1,*

$$\lim_{t\to\infty} \frac{B(t)}{t} \to 0.$$

Corollary 1.1. *The tail σ-field of Brownian motion is trivial.*

Proof. This follows from the fact that $\mathcal{F}_0^0 \equiv \mathcal{F}_0^+$, while $\mathcal{F}_0^0$ is trivial. $\square$